Exponentially small expansions of the Wright function on the Stokes lines

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Abstract

We investigate a particular aspect of the asymptotic expansion of the Wright function ${}_p\Psi_q(z)$ for large $|z|$. In the case $p = 1$, $q \geq 0$, we establish the form of the exponentially small expansion of this function on certain rays in the z-plane (known as Stokes lines). The importance of such exponentially small terms is encountered in Analytic Probability Theory and in the Theory of Generalised Linear Models. In addition, the transition of the Stokes multiplier connected with the subdominant exponential expansion across the Stokes lines is shown to obey the familiar error-function smoothing law expressed in terms of an appropriately scaled variable. Some numerical examples which confirm the accuracy of the expansion are given.

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1. Introduction

This is an asymptotic analysis paper dealing with new subtle properties of a particular class of special functions which arise in analytic Probability Theory. This paper has its outgrowth in the works of specialists such as Linnik [5], Skorokhod [18] and Zolotarev [27]; see also Ibragimov and Linnik [4] and Zolotarev [28]. All these publications considered asymptotic properties of the extreme stable laws and essentially employed the method of steepest descents among their machinery; see Section 3 for more detail. However, this last method has its own limitations and other elaborate procedures have been developed for the derivation of asymptotic representations of certain special functions. Some of these methods related to the so-called Wright function are reviewed in the first part of Section 2.

On the other hand, new trends in Distribution Theory, Analytical Probability and the Theory of Generalised Linear Models, which involve the construction of refined saddle-point approximations (some of which were considered in [20, 21]), encouraged the re-examination of the existing asymptotic properties of the Wright function. A refinement in the asymptotic structure of the Wright function on certain rays known as Stokes lines is more suitable for applications in probability and statistics. This is a central topic of this paper; see the second part of Section 2 for more detail.

We consider the Wright function (a generalised hypergeometric function) defined by

$$ {}_p\Psi_q(z) = \sum_{n=0}^{\infty} g(n)\,\frac{z^n}{n!}, \qquad g(n) = \frac{\prod_{r=1}^{p}\Gamma(\alpha_r n + a_r)}{\prod_{r=1}^{q}\Gamma(\beta_r n + b_r)}, \tag{1.1} $$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. We also assume throughout that the α_r and a_r are subject to the restriction

$$\alpha_r n + a_r \neq 0, -1, -2, \ldots \qquad (n = 0, 1, 2, \ldots\ ;\ 1 \leq r \leq p) \tag{1.2}$$

so that no gamma function in the numerator in (1.1) is singular. In the special case $\alpha_r = \beta_r = 1$, the function ${}_p\Psi_q(z)$ reduces to a multiple of the ordinary hypergeometric function ${}_pF_q((a)_p; (b)_q; z)$; see, for example, [19, p. 40].

We introduce the parameters associated with $g(n)$ given by

$$\kappa = 1 + \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r, \qquad h = \prod_{r=1}^{p} \alpha_r^{\alpha_r} \prod_{r=1}^{q} \beta_r^{-\beta_r},$$

$$\vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}(q-p). \tag{1.3}$$

If it is supposed that α_r and β_r are such that $\kappa > 0$ then ${}_p\Psi_q(z)$ is uniformly and absolutely convergent for all finite z. If $\kappa = 0$, the sum in (1.1) has a finite radius of convergence equal to h^{-1}, whereas for $\kappa < 0$ the sum is divergent for all nonzero values of z. The parameter κ will be found to play a critical role in the asymptotic theory of ${}_p\Psi_q(z)$ by determining the sectors in the z plane in which its behaviour is either exponentially large, algebraic or exponentially small in character as $|z| \to \infty$.

The determination of the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$ and finite values of the parameters has a long history; for details, see [16, §2.3]. Detailed investigations were carried out by Wright [25, 26] and by Braaksma [2] for a more general class of integral functions than (1.1). We present a summary of the expansion of ${}_p\Psi_q(z)$ when the parameter κ satisfies $0 < \kappa < 2$. In this case, the function has a composite expansion consisting of a single exponential expansion and, in general, p algebraic expansions.

When $0 < \kappa \leq 1$, which is the case that concerns us here, a Stokes phenomenon occurs on the rays $\arg z = \pm\pi\kappa$, where the exponential expansion is maximally subdominant with respect to the algebraic expansions. A more precise understanding of the asymptotic behaviour of ${}_p\Psi_q(z)$ in this case can be achieved by taking into account this phenomenon. Our principal aim in this paper is to examine the form of the exponentially small expansion on the Stokes lines and to supply numerical verifications. We carry this out only for the case $p = 1$, $q \geq 0$; the case $p \geq 1$, $q = 0$ and $\alpha_r = \alpha$ $(1 \leq r \leq p)$ has been discussed in [14]. As a by-product, we also demonstrate that the so-called Stokes multiplier associated with the exponential expansion obeys the familiar error-function smoothing law, originally due to Berry [1]. This smooth transition has been established for the generalised Bessel function ($p = 0$, $q = 1$) in [23] and for the Mittag-Leffler function ($p = q = 1$ with $\alpha_1 = a_1 = 1$) in [11, 24].

The structure of the paper is as follows. In Section 2 we review the main asymptotic properties of the Wright function. The classical example for which we compare our approach with that given in Ibragimov and Linnik [4, p. 62] is discussed in Section 3. Section 4 contains the main asymptotic discussion of ${}_1\Psi_q(z)$, with some numerical examples confirming the accuracy of the expansions given in Section 5. An example, which pertains to an important class of Poisson mixtures, is considered in the final part of Section 5. Some analytical details, mainly concerning the so-called terminant function used in the asymptotic treatment on the Stokes lines, are presented in the appendices.

2. Standard asymptotic theory of ${}_p\Psi_q(z)$ for $|z| \to \infty$

We present the standard asymptotic expansion of the integral function ${}_p\Psi_q(z)$ as $|z|\to\infty$ for $0<\kappa<2$ and finite values of the parameters given in [26] and [2]; see also [17, §2.3]. In order to give this expansion we first need to define the associated exponential and algebraic expansions $E_{p,q}(z)$ and $H_{p,q}(z)$ respectively, together with a lemma on the inverse-factorial expansion of the ratio of products of gamma functions. Throughout we let ϵ denote an arbitrarily small positive quantity.

By application of Stirling's formula for the gamma function we have the important lemma [2, §3], [17, p. 39]

Lemma 1 *Let M denote a positive integer and suppose that $\kappa>0$. Then there exist coefficients A_j $(0\leq j\leq M-1)$ such that*

$$\Gamma(s)\,\frac{\prod_{r=1}^q\Gamma(1-b_r+\beta_r s)}{\prod_{r=1}^p\Gamma(1-a_r+\alpha_r s)}$$
$$=\frac{\kappa(h\kappa^\kappa)^{-s}}{(2\pi)^{p-q}}\left\{\sum_{j=0}^{M-1}(-)^jA_j\Gamma(\kappa s+\vartheta-j)+\sigma_M(s)\Gamma(\kappa s+\vartheta-M)\right\},\qquad(2.1)$$

where the parameters κ, h and ϑ have the values given in (1.3). The remainder function $\sigma_M(s)$ is analytic in s except at the poles of the corresponding gamma function ratio and is such that $\sigma_M(s)=O(1)$ as $|s|\to\infty$ uniformly in $|\arg s|\leq\pi-\epsilon$.

The coefficients A_j are independent of s and depend only on the parameters p, q, α_r, β_r, a_r and b_r. An algorithm for their numerical evaluation is described in [12]. The leading coefficients A_0 and A_1 are specified by [12]

$$A_0=(2\pi)^{\frac{1}{2}(p-q)}\kappa^{-\frac{1}{2}-\vartheta}\prod_{r=1}^p\alpha_r^{a_r-\frac{1}{2}}\prod_{r=1}^q\beta_r^{\frac{1}{2}-b_r},\qquad A_1=\tfrac{1}{2}\kappa A_0(\mathcal{A}+\tfrac{1}{6}\mathcal{B}),\qquad(2.2)$$

where

$$\mathcal{A}=\sum_{r=1}^p\frac{a_r(a_r-1)}{\alpha_r}-\sum_{r=1}^q\frac{b_r(b_r-1)}{\beta_r}-\frac{\vartheta(1-\vartheta)}{\kappa},\qquad \mathcal{B}=\sum_{r=1}^p\frac{1}{\alpha_r}-\sum_{r=1}^q\frac{1}{\beta_r}+\frac{1-\kappa}{\kappa}.$$

Then the exponential expansion $E_{p,q}(z)$ is given by the formal asymptotic sum

$$E_{p,q}(z):=Z^\vartheta e^Z\sum_{j=0}^\infty A_jZ^{-j},\qquad Z=\kappa(hz)^{1/\kappa},\qquad(2.3)$$

where the coefficients A_j are those appearing in the inverse factorial expansion in (2.1).

The algebraic expansion $H_{p,q}(z)$ follows from the Mellin-Barnes integral representation [17, §2.4]

$${}_p\Psi_q(z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(s)g(-s)(ze^{\mp\pi i})^{-s}ds,\qquad|\arg(-z)|<\tfrac{1}{2}\pi(2-\kappa)\qquad(2.4)$$

where the path of integration is indented near $s=0$ to separate[1] the poles of $\Gamma(s)$ from those of $g(-s)$ situated at

$$s=(a_r+k)/\alpha_r,\qquad k=0,1,2,\ldots\ (1\leq r\leq p).\qquad(2.5)$$

In general there will be p such sequences of simple poles though, depending on the values of α_r and a_r, some of these poles could be multiple poles or even ordinary points if any of the $\Gamma(\beta_r s+b_r)$ are singular there. Displacement of the contour to the right over the

[1]This is always possible when the condition (1.2) is satisfied.

poles of $g(-s)$ then yields the algebraic expansion of ${}_p\Psi_q(z)$ valid in the sector in (2.4). If it is assumed that the parameters are such that the poles in (2.5) are all simple we obtain the algebraic expansion given by $H_{p,q}(ze^{\mp\pi i})$, where

$$H_{p,q}(z) := \sum_{n=1}^{p} \alpha_n^{-1} z^{-a_n/\alpha_n} S_{p,q}(z;n) \tag{2.6}$$

and $S_{p,q}(z;n)$ denotes the formal asymptotic sum

$$S_{p,q}(z;n) := \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \Gamma\left(\frac{k+a_n}{\alpha_n}\right) \frac{\prod_{r=1}^{\prime\, p} \Gamma(a_r - \alpha_r(k+a_n)/\alpha_n)}{\prod_{r=1}^{q} \Gamma(b_r - \beta_r(k+a_n)/\alpha_n)} z^{-k/\alpha_n}, \tag{2.7}$$

with the prime indicating the omission of the term corresponding to $r = n$ in the product. The expression in (2.6) consists of (at most) p expansions each with the leading behaviour z^{-a_n/α_n} $(1 \leq n \leq p)$. When the parameters α_r and a_r are such that some of the poles are of higher order, the expansion (2.7) is invalid and the residues must then be evaluated according to the multiplicity of the poles concerned; this will lead to terms involving $\log z$ in the algebraic expansion.

The expansion theorem for ${}_p\Psi_q(z)$ is then as follows.

Theorem 1 *If $0 < \kappa < 2$, then*

$${}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \leq \frac{1}{2}\pi\kappa \\[2ex] H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg(-z)| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon \end{cases} \tag{2.8}$$

as $|z| \to \infty$. The upper or lower sign in $H_{p,q}(ze^{\mp\pi i})$ is chosen according as z lies in the upper or lower half-plane, respectively.

This result gives the dominant expansion of ${}_p\Psi_q(z)$. It is seen that the z-plane is divided into two sectors, with a common vertex at $z = 0$, by the rays (the anti-Stokes lines) $\arg z = \pm\frac{1}{2}\pi\kappa$. In the sector $|\arg z| < \frac{1}{2}\pi\kappa$, the asymptotic character of ${}_p\Psi_q(z)$ is exponentially large whereas in the complementary sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$, the dominant expansion of ${}_p\Psi_q(z)$ is algebraic in character.

We now confine our attention in the rest of this paper to the case $0 < \kappa \leq 1$, where we shall endeavour to provide a more refined asymptotic description of ${}_p\Psi_q(z)$. The exponential expansion $E_{p,q}(z)$ is still present beyond the sector $|\arg z| < \frac{1}{2}\pi\kappa$ where it becomes subdominant in the sectors $\frac{1}{2}\pi\kappa < |\arg z| < \pi\kappa$. The rays $\arg z = \pm\pi\kappa$, where $E_{p,q}(z)$ is *maximally* subdominant with respect to $H_{p,q}(ze^{\mp\pi i})$, are called Stokes lines.[2] As these rays are crossed (in the sense of increasing $|\arg z|$) the exponential expansion should switch off according to the now familiar error-function smoothing law [1]. In view of this interpretation of the Stokes phenomenon a more precise version of Theorem 1 in the case $0 < \kappa < 1$ is the following:

Conjecture 1 *If $0 < \kappa < 1$, then*

$${}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \leq \pi\kappa - \epsilon \\[2ex] \frac{1}{2}\mathcal{E}_{p,q}(z) + H^o_{p,q}(ze^{\mp\pi i}) & \text{on} \quad \arg z = \pm\pi\kappa \\[2ex] H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg(-z)| \leq \pi(1-\kappa) - \epsilon \end{cases} \tag{2.9}$$

[2] The positive real axis $\arg z = 0$ is also a Stokes line where the algebraic expansion is maximally subdominant.

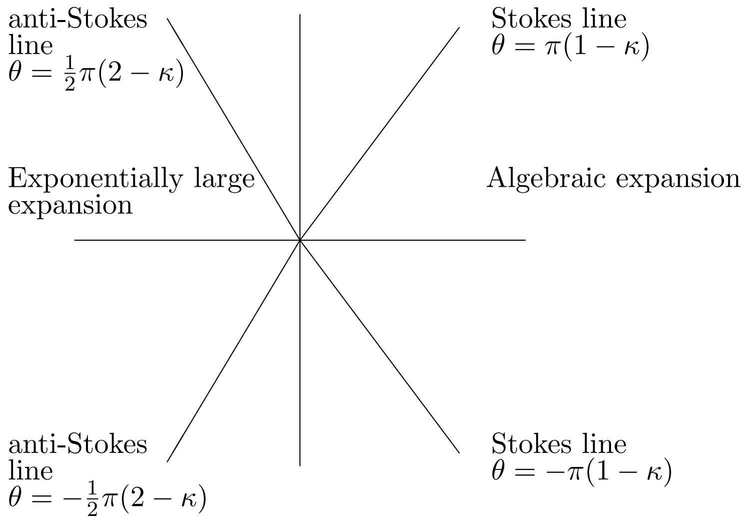

Figure 1: The exponentially large and algebraic sectors associated with ${}_p\Psi_q(-z)$ in the complex z-plane when $0<\kappa<1$ and $\theta = \arg z$. Between the Stokes and anti-Stokes lines the expansion consists of an algebraic expansion and a subdominant exponential expansion. On the anti-Stokes lines the algebraic and exponential expansions are of comparable significance.

as $|z|\to\infty$. *The upper or lower sign in* $H_{p,q}(ze^{\mp\pi i})$ *is chosen according as* z *lies in the upper or lower half-plane, respectively. In the middle expression on the Stokes lines* $\arg z = \pm\pi\kappa$, *the superscript o denotes that the algebraic expansions in* $H_{p,q}(ze^{\mp\pi i})$ *are optimally truncated at, or near, the terms of least magnitude. The expansion* $\mathcal{E}_{p,q}(z)$ *denotes the expansion* $E_{p,q}(z)$ *augmented by the presence of a series of the form* $2|Z|^{\vartheta}e^{-|Z|}\sum B_j|Z|^{-j-1/2}$, *where* Z *is defined in (2.3) and the coefficients* B_j *depend on the various parameters and the coefficients* A_j.

The exponentially small expansion on $\arg z = \pm\pi$ in the case $\kappa = 1$ is considered in Section 4.3. Thus, although the expansion in (2.8) is a valid asymptotic description of ${}_p\Psi_q(z)$, more acurate evaluation will result from taking into account the Stokes phenomenon as certain rays are crossed. It is our aim here to establish (2.9) for the case $p=1$, $q\geq 0$. A particular case of (2.9) when $p\geq 1$ and $q=0$ was considered in [10].

In our analysis of the case $0<\kappa<1$ in Sections 4.1 and 4.2, we shall find it more convenient to deal with the function ${}_p\Psi_q(-z)$. From (2.9), we then have the expansion

$$ {}_p\Psi_q(-z) \sim \begin{cases} H_{p,q}(z) & \text{in} \quad |\arg z| \leq \pi(1-\kappa)-\epsilon \\[2ex] \frac{1}{2}\mathcal{E}_{p,q}(ze^{-\pi i}) + H^o_{p,q}(z) & \text{on} \quad \arg z = \pi(1-\kappa) \\[2ex] E_{p,q}(ze^{-\pi i}) + H_{p,q}(z) & \text{in} \quad \pi \leq \arg z \leq \pi(1-\kappa)+\epsilon. \end{cases} \tag{2.10} $$

A similar expansion with $e^{-\pi i}$ replaced by $e^{\pi i}$ in the exponential series holds in the conjugate sector $[-\pi, -\pi(1-\kappa)]$. The division of the z-plane into regions where ${}_p\Psi_q(-z)$ has either algebraic or exponentially large behaviour for large $|z|$ is illustrated in Fig. 1. The rays $\arg z = \pm\pi(1-\kappa)$ are Stokes lines, on which the exponential series in (2.10) is maximally subdominant relative to the algebraic series. As $|\arg z|$ increases, the dominance of the algebraic over the exponential series progressively diminishes until on $\arg z = \pm\frac{1}{2}\pi(2-\kappa)$ — the anti-Stokes lines — there is an exchange of dominance: in the sector $|\arg(-z)| < \frac{1}{2}\pi\kappa$, the exponential expansion dominates the algebraic expansion.

3. An example related to stable probability laws

An example where the exponentially small expansion of a particular case of the Wright function on a Stokes line is of importance has been encountered in a probabilistic context by Ibragimov and Linnik [4, p. 62, Thm. 2.4.6]. They required the expansion as $x \to +\infty$ of the function

$$\mathcal{F}(x) := \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-n\alpha)} \qquad (0 < \alpha < 1).$$

Note in passing that this function is closely related to the probability density function of a positive stable distribution with index α; see [20] for more detail. By means of the reflection formula for the gamma function this can be written as

$$\mathcal{F}(x) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \Gamma(n\alpha + 1) \sin \pi n\alpha = \frac{1}{2\pi i} \{{}_1\Psi_0(xe^{\pi i\kappa}) - {}_1\Psi_0(xe^{-\pi i\kappa})\}, \tag{3.1}$$

where the Wright functions with $p = 1$, $q = 0$ are associated with the parameters $\kappa = 1-\alpha$ and $a = 1$. It is seen that the asymptotics of $\mathcal{F}(x)$ as $x \to +\infty$ results from the expansion of ${}_1\Psi_0(z)$ on the Stokes lines $\arg z = \pm\pi\kappa$. The above-mentioned authors did not employ the asymptotic theory of the Wright function, however, but based their investigation on the following Laplace integral representation[3]

$$\mathcal{F}(x) = \frac{1}{\pi} \,\mathrm{Im} \int_0^{\infty} \exp\,[-t - xt^{\alpha}e^{-\pi i\alpha}]\, dt \tag{3.2}$$

combined with an application of the method of steepest descents.

To illustrate, let us consider

$${}_1\Psi_0(xe^{\pi i\kappa}) = \sum_{n=0}^{\infty} \frac{(xe^{\pi i\kappa})^n}{n!} \Gamma(n\alpha + 1) = \int_0^{\infty} \exp\,[-t - xt^{\alpha}e^{-\pi i\alpha}]\, dt$$

for $x \to +\infty$, where the parameter $a = 1$. Making the substitution $t = \tau x^{1/\kappa}$, we then find

$${}_1\Psi_0(xe^{\pi i\kappa}) = x^{1/\kappa} \int_0^{\infty} \exp\,[x^{1/\kappa}\phi(\tau)]\, d\tau, \qquad \phi(\tau) := -\tau - \tau^{\alpha}e^{-\pi i\alpha}. \tag{3.3}$$

The integrand has a saddle point (where $\phi'(\tau) = 0$) at $\tau_s = -\alpha^{1/\kappa}$. We can deform the integration path into the segement $[0, \tau_s]$ on the upper side of the branch cut on the negative τ-axis and thence along the steepest descent path C in the upper half-plane that emanates from τ_s parallel to the imaginary axis and passes to infinity in the first quadrant; see Fig. 2. The integral along the negative τ-axis produces the algebraic expansion and, with $\tau = e^{\pi i}u$ in the integral in (3.3) over the interval $0 \le u \le u_s$, $u_s = \alpha^{1/\kappa}$, this becomes

$$-x^{1/\kappa} \int_0^{u_s} \exp\,[x^{1/\kappa}(u - u^{\alpha})]\, du \sim -\frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{\Gamma((k+1)/\alpha)}{k!}\, x^{-(k+1)/\alpha} \qquad (x \to +\infty) \tag{3.4}$$

upon expansion of the factor $\exp\,[ux^{1/\kappa}]$ as a Maclaurin series and replacement of the upper limit of integration by infinity.

Figure 2: The steepest descent path C in the upper half-plane through the saddle τ_s.

[3] This is easily established by writing $\exp[-xt^{\alpha}e^{-\pi i\alpha}]$ as a Maclaurin series followed by term-by-term integration. In [4] the integration path was rotated through $-\pi/2$.

The contribution from the steepest descent path C as $x \to +\infty$ is obtained by a routine application of the method of steepest descents; see, for example, [6, §7.1]. As $x \to +\infty$, we have

$$x^{1/\kappa}\int_C \exp[x^{1/\kappa}\phi(\tau)]\,d\tau \sim x^{1/\kappa}\exp[x^{1/\kappa}\phi(\tau_s)]\sum_{k=0}^{\infty}\frac{c_k\Gamma(\frac{1}{2}k+\frac{1}{2})}{x^{(k+1)/(2\kappa)}}, \tag{3.5}$$

where $\phi(\tau_s) = -\kappa\alpha^{\alpha/\kappa}$ and the coefficients c_k are given by [3, p. 119], [13, p. 13]

$$c_0 = \frac{1}{(-2\phi'')^{1/2}} = \frac{i\alpha^{1/(2\kappa)}}{(2\kappa)^{\frac{1}{2}}}, \qquad c_1 = \frac{\phi'''}{3\phi''^2} = -\frac{\alpha-2}{3\kappa},$$

$$c_2 = -\frac{c_0}{12\phi''}\left(\frac{5\phi'''^2}{\phi''^2} - \frac{3\phi^{iv}}{\phi''}\right) = -\frac{c_0(\alpha-2)(2\alpha-1)}{12\kappa\alpha^{1/\kappa}},$$

$$c_3 = -\frac{1}{3\phi''^2}\left(\frac{8\phi'''^3}{9\phi''^3} - \frac{\phi'''\phi^{iv}}{\phi''^2} + \frac{\phi^v}{5\phi''}\right) = \frac{2(\alpha-2)(1-\alpha-2\alpha^2)}{135\kappa^2\alpha^{1/\kappa}},$$

$$c_4 = \frac{35c_0}{36\phi''^2}\left(\frac{11\phi'''^4}{24\phi''^4} - \frac{3}{4}\left(\frac{\phi'''^2}{\phi''^2} - \frac{\phi^{iv}}{6\phi''}\right)\frac{\phi^{iv}}{\phi''} + \frac{\phi'''\phi^v}{5\phi''^2} - \frac{\phi^{vi}}{35\phi''}\right)$$
$$= \frac{c_0(\alpha-2)(2\alpha-1)}{864\kappa^2\alpha^{2/\kappa}}(2+19\alpha+2\alpha^2), \ldots$$

with the derivatives of $\phi(\tau)$ being evaluated at $\tau = \tau_s$. Here we have used the fact that $\phi^{(n+1)}(\tau_s) = (1-\alpha)_n\alpha^{-n/\kappa}$ for integer $n \geq 1$, where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

In terms of the notation in (1.3), where

$$\kappa = 1-\alpha, \quad h = \alpha^\alpha, \quad \vartheta = \tfrac{1}{2}, \quad A_0 = (2\pi\alpha)^{1/2}/\kappa, \quad X = \kappa(hx)^{1/\kappa},$$

we then obtain the expansion of ${}_1\Psi_0(xe^{\pi i\kappa})$ from (3.4) and (3.5) in the form

$${}_1\Psi_0(xe^{\pi i\kappa}) \sim -\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{\Gamma((k+1)/\alpha)}{k!}x^{-(k+1)/\alpha} + \frac{1}{2}iA_0X^{1/2}e^{-X}\sum_{k=0}^{\infty}\frac{\hat{c}_k\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2})X^{k/2}}, \tag{3.6}$$

as $x \to +\infty$, where we have defined the coefficients

$$\hat{c}_k := (c_k/c_0)(\kappa h^{1/\kappa})^{k/2}.$$

It is understood that the dominant algebraic expansion in (3.6) has to be optimally truncated at, or near, the term of least magnitude so that the exponentially small terms are of numerical significance. Then (3.6) can be seen to be of the form of the expansion on the Stokes line given in (2.9) in the case $p = 1$, $q = 0$. We remark, however, that (3.6) does not capture the exponentially small terms completely, as we shall demonstrate in Section 5 using a Mellin-Barnes integral treatment. The reason for this deficiency is due to the inability of the algebraic expansion in (3.4) to account for the exponentially small contribution resulting from *the approach to the saddle point* on the real τ-axis.

For the determination of the expansion of $\mathcal{F}(x)$ in (3.1) as $x \to +\infty$, this last fact is immaterial since the integral along the negative τ-axis is real and so makes no contribution to (3.2). Similarly, the exponentially small terms in (3.5) corresponding to odd index k make no contribution since they are all real. We then obtain the exponentially small expansion of $\mathcal{F}(x)$ in the form

$$\mathcal{F}(x) \sim \frac{1}{2\pi}X^{1/2}e^{-X}\sum_{k=0}^{\infty}(-)^kA_kX^{-k} \qquad (x \to +\infty), \tag{3.7}$$

where

$$A_k := A_0\,\hat{c}_{2k}(\tfrac{1}{2})_k = A_0\,(c_{2k}/c_0)\,(\tfrac{1}{2})_k(\kappa h^{1/\kappa})^k.$$

This expansion is equivalent to that obtained by Ibragimov and Linnik [4, Eq. (2.4.42)]; see also [28, p. 62] and [20, Eq. (4.15)]. The coefficents $(-\alpha)^k A_k$ have been called the Zolotarev polynomials[4] in [20], where the first five polynomials are displayed; see also [22]. The explicit representation of the coefficients A_k for $k \leq 3$ is

$$A_1/A_0 = \frac{(\alpha-2)(2\alpha-1)}{24\alpha}, \qquad A_2/A_0 = \frac{(\alpha-2)(2\alpha-1)}{1152\alpha^2}\,(2+19\alpha+2\alpha^2),$$

$$A_3/A_0 = -\frac{(\alpha-2)(2\alpha-1)}{414720\alpha^3}\,(556\alpha^4-1628\alpha^3-9093\alpha^2-1628\alpha+556), \tag{3.8}$$

with $A_0 = (2\pi\alpha)^{1/2}/\kappa$.

The treatment of ${}_p\Psi_0(z)$ with $p \geq 2$ by this integral approach, however, becomes more difficult as the Wright function is then given by a p-dimensional Laplace integral; see [16, p. 133].

4. The case $p = 1$, $q \geq 0$

In this section we give a proof of the statement in Conjecture 1 for the particular case $p = 1$ ($\alpha_1 = \alpha$, $a_1 = a$) and $q \geq 0$. We consider the function

$${}_1\Psi_q(-z) = \sum_{n=0}^\infty \frac{\Gamma(\alpha n + a)}{\prod_{r=1}^q \Gamma(\beta_r n + b_r)}\,\frac{(-z)^n}{n!},$$

which, from (1.3), is associated with the parameters

$$\kappa = 1 + \Sigma\,\beta_r - \alpha, \qquad \vartheta = a - \Sigma\,b_r + \tfrac{1}{2}(q-1), \qquad h = \alpha^\alpha \prod_{r=1}^q \beta_r^{-\beta_r}. \tag{4.1}$$

When $q = 0$ the product of gamma functions in the denominator is replaced by unity and the parameter α is then restricted to satisfy $0 < \alpha < 1$; when $q \geq 1$, the parameters can be chosen so that $0 < \kappa \leq 1$.

Throughout this section we write $\Sigma\,\beta_r$ and $\Sigma\,b_r$ where it is understood that the summation index r runs from 1 to q; in the case $q = 0$ these sums are interpreted as zero. Our attention will be confined to the Stokes line $\arg z = \pi(1-\kappa)$ in the upper half-plane, across which in the sense of increasing $\arg z$ the subdominant exponential series is switched on. The treatment of the Stokes line $\arg z = -\pi(1-\kappa)$ in the lower half-plane is similar. When $\kappa = 1$, the Stokes line is the positive real axis. In what follows we shall write

$$Z = \kappa(hz)^{1/\kappa}, \qquad X = \kappa(h|z|)^{1/\kappa}, \qquad \mu = \frac{\kappa}{\alpha}, \qquad \nu = \vartheta + \mu(a+m), \tag{4.2}$$

where m denotes a positive integer to be specified below.

4.1 Expansion of ${}_1\Psi_q(-z)$ near the Stokes line $\arg z = \pi(1-\kappa)$

Our starting point is the integral representation obtained from (2.4) subject to condition (1.2)

$${}_1\Psi_q(-z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s)\,\frac{\Gamma(-\alpha s + a)}{\prod_{r=1}^q \Gamma(-\beta_r s + b_r)}\,z^{-s}\,ds \qquad (|\arg z| < \tfrac{1}{2}\pi(2-\kappa)), \tag{4.3}$$

[4] It has recently been discovered that the same name is already in use for a completely different set of polynomials named after the 19th century Russian mathematician E. I. Zolotarev.

where the integration path is indented to separate the two sequences of poles of the integrand. It is seen that the sector of validity of (4.3) encloses the Stokes lines $\arg z = \pm\pi(1-\kappa)$. Displacement of the integration path to the right in the usual manner over the first m poles of $\Gamma(-\alpha s+a)$ situated at $s=(k+a)/\alpha$ $(k=0,1,2,\ldots)$ then produces

$$ {}_1\Psi_q(-z)=\frac{1}{\alpha}\sum_{k=0}^{m-1}\frac{(-)^k}{k!}\,\frac{\Gamma((k+a)/\alpha)z^{-(k+a)/\alpha}}{\prod_{r=1}^q\Gamma(b_r-\beta_r(a+k)/\alpha)}+P_m(z), \tag{4.4} $$

where, upon use of the reflection formula for the gamma function,

$$ P_m(z)=\frac{(-\pi)^{1-q}}{2\pi i}\int_{L_m}\frac{G(s)\,\Xi(s)}{\sin\pi(\alpha s-a)}\,z^{-s}ds \tag{4.5} $$

with

$$ G(s)=\Gamma(s)\,\frac{\prod_{r=1}^q\Gamma(1-b_r+\beta_r s)}{\Gamma(1-a+\alpha s)},\qquad \Xi(s)=\prod_{r=1}^q\sin\pi(\beta_r s-b_r). $$

The path L_m denotes a path (possibly indented) parallel to the imaginary s-axis with $\mathrm{Re}(s)=(a+m-c)/\alpha$, $0<c<1$. We recognise that the series in (4.4) is the first m terms of the algebraic expansion $H_{1,q}(z)$ in (2.6) and (2.7).

In order to detect the appearance of the exponential series as $|z|\to\infty$ we need to choose the index m in the algebraic expansion (4.4) to correspond to optimal truncation k_0; that is at, or near, the least term in magnitude. We make use of the well-known result $\Gamma(k+a)/\Gamma(k+b)\sim k^{a-b}$ for $k\to+\infty$ and that for the asymptotic series $\sum u_k z^{-k}$ the least term arises when $|z|\simeq|u_{k+1}/u_k|$. We then obtain the least term with index k_0 given by

$$ |z|^{1/\alpha}\simeq\frac{\Gamma((k_0+a+1)/\alpha)}{k_0\Gamma((k_0+a)/\alpha)}\prod_{r=1}^q\frac{\Gamma(1-b_r+\beta_r(k_0+a+1)/\alpha)}{\Gamma(1-b_r+\beta_r(k_0+a)/\alpha)} $$
$$ \simeq\frac{1}{\alpha}\left(\frac{k_0}{\alpha}\right)^{-1+1/\alpha}\prod_{r=1}^q\left(\frac{k_0\beta_r}{\alpha}\right)^{\beta_r/\alpha}=\left(\frac{k_0}{\alpha}\right)^{\kappa/\alpha}h^{-1/\alpha}, $$

whence

$$ \mu k_0\simeq X, \tag{4.6} $$

where μ and X are defined in (4.2). We now set $m=k_0$ in (4.4) and so consider the algebraic expansion $H_{1,q}(z)$ to be *optimally truncated*.

We make use of the expansion for the product of sines appearing in $\Xi(s)$ given by

$$ \Xi(s)=(-2i)^{-q}e^{-\pi i(\Sigma\,\beta_r s-\Sigma\,b_r)}\sum_{n=1}^{2q}\Omega_n e^{\pi i\delta_n s},\qquad \Omega_1=1,\ \Omega_{2q}=(-)^q e^{-2\pi i\Sigma b_r}, \tag{4.7} $$

where the quantities δ_n satisfy

$$ 0=\delta_1<\delta_2\leq\delta_3\leq\ldots\leq\delta_{2q-1}<\delta_{2q}=2\Sigma\beta_r $$

and Ω_n are coefficients such that $|\Omega_n|=1$ when b_r $(1\leq r\leq q)$ are real. The precise values of δ_n and Ω_n $(2\leq n\leq 2q-1)$ will not be necessary in the following treatment. Then, from (4.5),

$$ P_m(z)=\sum_{n=1}^{2q}\Omega_n R_m(Z_n),\qquad Z_n:=\kappa(hz_n)^{1/\kappa},\quad z_n:=ze^{\pi i(\Sigma\beta_r-\delta_n)}, \tag{4.8} $$

where

$$ R_m(Z_n)=\frac{(2\pi)^{1-q}}{4\pi}e^{\pi i(a-\vartheta)}\int_{L_m}\frac{(h\kappa^\kappa)^s G(s)\,Z_n^{-\kappa s}}{\sin\pi(\alpha s-a)}\,ds. $$

When m is chosen to be the optimal truncation index, it follows from (4.6) that $m \to \infty$ as $|z| \to \infty$, so that on the integration path L_m in (4.5) we have $|s|$ everywhere large. Consequently, we may employ the expansion (2.1) for $G(s)$ in the form

$$G(s) = \frac{\kappa(h\kappa^\kappa)^{-s}}{(2\pi)^{1-q}} \left\{ \sum_{j=0}^{M-1} (-)^j A_j \Gamma(\kappa s + \vartheta - j) + \sigma_M(s)\Gamma(\kappa s + \vartheta - M) \right\}$$

for positive integer M, where $\sigma_M(s) = O(1)$ as $|s| \to \infty$ in $|\arg s| < \pi$. Then

$$R_m(Z_n) = \frac{\mu e^{-\pi i\vartheta}}{4\pi}(Z_n e^{-\pi i/\mu})^{-\mu(a+m)} \sum_{j=0}^{M-1} (-)^j A_j \int_{-c-\infty i}^{-c+\infty i} \frac{\Gamma(\mu s + \nu - j)}{\sin \pi s} Z_n^{-\mu s} ds + R_{M,m}(Z_n), \tag{4.9}$$

where $0 < c < 1$ and in the integrals appearing in the sum over j we have replaced the variable s by $(s + a + m)/\alpha$ with the introduction of the parameter ν defined in (4.2). The remainder term $R_{M,m}(Z_n)$ is given by

$$R_{M,m}(Z_n) = \frac{\kappa}{4\pi} e^{\pi i(a-\vartheta)} Z_n^{-\mu a} \int_{-c+m-\infty i}^{-c+m+\infty i} \frac{\Gamma(\mu s + \mu a + \vartheta - M)}{\sin \pi s} \hat{\sigma}_M(s) Z_n^{-\mu s} ds, \tag{4.10}$$

where the function $\hat{\sigma}_M(s)$ denotes $\sigma_M(s)$ with s replaced by $(s + a)/\alpha$.

We now let $\theta = \arg z$ and introduce $\omega(\theta)$ as the measure of the angular separation from the Stokes line $\arg z = \pi(1 - \kappa)$ by

$$\omega(\theta) := \kappa^{-1}\{\theta - \pi(1 - \kappa)\}. \tag{4.11}$$

It then follows that

$$\arg Z_n = \frac{\theta}{\kappa} + \frac{\pi}{\kappa}(\Sigma\beta_r - \delta_n) = \omega(\theta) + \frac{\pi}{\mu}\left(1 - \frac{\delta_n}{\alpha}\right) \tag{4.12}$$

upon use of the definition of κ in (4.1). The sectors of validity of the integrals in (4.9) are $|\arg Z_n| < \frac{1}{2}\pi + \pi/\mu$ and, for $1 \le n \le 2q$, these form a series of overlapping sectors. It is easy to verify that they have the common sector of validity $|\arg Z| < \pi(2-\kappa)/(2\kappa)$, which corresponds to the sector in (4.3), and that each sector contains the Stokes line $\omega(\theta) = 0$.

If we now introduce the terminant function $T_\nu(\mu; x)$ defined in (A.1) and (A.2) by

$$T_\nu(\mu; x) = (xe^{-\pi i/\mu})^{-\nu} \exp\,[xe^{-\pi i/\mu}] \, \frac{\mu}{4\pi} \int_{-c-\infty i}^{-c+\infty i} \Gamma(\mu s + \nu) \, \frac{x^{-\mu s}}{\sin \pi s} \, ds$$

when $|\arg x| < \frac{1}{2}\pi + \pi/\mu$ and $0 < c < 1$, then

$$\begin{aligned} R_m(Z_n) \;&=\; e^{-\pi i\vartheta}(Z_n e^{-\pi i/\mu})^\vartheta e^{-Z_n e^{-\pi i/\mu}} \sum_{j=0}^{M-1} (-)^j A_j (Z_n e^{-\pi i/\mu})^{-j} \, T_{\nu-j}(\mu; Z_n) + R_{M,m}(Z_n) \\ &=\; (Ze^{-i\psi_n})^\vartheta e^{Ze^{-i\psi_n}} \sum_{j=0}^{M-1} A_j (Ze^{-i\psi_n})^{-j} \, T_{\nu-j}(\mu; Z_n) + R_{M,m}(Z_n), \end{aligned} \tag{4.13}$$

where, from (4.12), we have employed

$$Z_n e^{-\pi i/\mu} = Ze^{\pi i - i\psi_n}, \qquad \psi_n := \frac{\pi}{\kappa}(1 + \delta_n)$$

and we recall that the m-dependence is contained in the parameter ν.

Bounds on the remainder $R_{M,m}(Z_n)$ in the expansion of $R_m(Z_n)$ truncated after M terms can be derived by application of the arguments given in [9]; see also [17, p. 244].

This is discussed in Appendix B, where it is shown that as $X \to \infty$, with $\nu \sim X$ by (4.6) and $\Delta_n := \pi\delta_n/\kappa$ $(1 \le n \le 2q)$,

$$\left.\begin{array}{lll} R_{M,m}(Z_n) & = O(X^{\vartheta-M}e^{-X}) & \text{in} \quad -2\pi/\mu + \Delta_n \le \omega(\theta) \le \Delta_n, \\ R_{M,m}(Z_1) & = O(X^{\vartheta-M}e^{Ze^{-\pi i/\kappa}}) & \text{in} \quad 0 < \omega(\theta) < \min\{\pi, 2\pi/\mu\}, \end{array}\right\}. \tag{4.14}$$

Now $\Delta_1 = 0$ and $\Delta_{2q} = 2\pi\Sigma\beta_r/\kappa \le 2\pi/\mu$, since $\alpha \ge \Sigma\beta_r$ when $\kappa \le 1$. When $\kappa < 1$, the sectors in the first line of (4.14) for $2 \le n \le 2q$ therefore contain the Stokes line $\omega(\theta) = 0$ in their interiors, whereas the sector corresponding to $n = 1$, viz. $-2\pi/\mu \le \omega(\theta) \le 0$, has a boundary on the Stokes line. When $\kappa = 1$, the sector corresponding to $n = 2q$, viz. $0 \le \omega(\theta) \le 2\pi/\mu$, also has a boundary on the Stokes line. The bound on $R_{M,m}(Z_1)$ beyond the Stokes line $\omega(\theta) > 0$ is given in the second line of (4.14).

From the large-variable asymptotic behaviour of the terminant function given in (A.11) and the fact that $\nu \sim X$, we see that in the neighbourhood of the Stokes line when $0 < \kappa < 1$

$$T_{\nu-j}(\mu; Z_n) \sim \begin{cases} \frac{1}{2} + \frac{1}{2}\text{erf}\,[\omega(\theta)\,(\frac{1}{2}X)^{1/2}] & (n = 1) \\ \\ e^{-X-Z_n e^{-i\psi_n}}O(X^{-1/2}) & (2 \le n \le 2q) \end{cases}$$

for finite values of the summation index j. The leading behaviour of $P_m(z)$ in the neighbourhood of the Stokes line $\theta = \pi(1-\kappa)$ when $\kappa < 1$ is therefore given by the term corresponding to $n = 1$ in (4.8), the other terms corresponding to $2 \le n \le 2q$ being exponentially small in this neighbourhood. Recalling that $\psi_1 = \pi/\kappa$ and $\Omega_1 = 1$, we therefore find from (4.8) and (4.13) that

$$\begin{aligned} P_m(z) &\sim (Ze^{-\pi i/\kappa})^{\vartheta} e^{Ze^{-\pi i/\kappa}} \sum_{j=0}^{\infty} A_j (Ze^{-\pi i/\kappa})^{-j}\, T_{\nu-j}(\mu; Z_1) \\ &\simeq \left\{\tfrac{1}{2} + \tfrac{1}{2}\text{erf}\,[\omega(\theta)(\tfrac{1}{2}X)^{1/2}]\right\} (Ze^{-\pi i/\kappa})^{\vartheta} e^{Ze^{-\pi i/\kappa}} \sum_{j=0}^{\infty} A_j (Ze^{-\pi i/\kappa})^{-j}. \end{aligned} \tag{4.15}$$

The result in (4.15) confirms the smooth switch-on of the exponentially small expansion in ${}_1\Psi_q(-z)$ as arg z increases across the Stokes line arg $z = \pi(1-\kappa)$. This is controlled by the multiplier

$$\tfrac{1}{2} + \tfrac{1}{2}\text{erf}\,[\omega(\theta)(\tfrac{1}{2}X)^{1/2}], \tag{4.16}$$

which changes from approximately zero when $\omega(\theta) < 0$ to approximately 1 when $\omega(\theta) > 0$ for large X within a zone centred on $\omega(\theta) = 0$ of angular width $O(X^{-1/2})$. On the Stokes line, $\omega(\theta) = 0$ and we find $P_m(z) \sim \frac{1}{2}E_{1,q}(ze^{-\pi i})$ to leading order, upon identification of the expansion in (4.15) as $E_{1,q}(ze^{-\pi i})$ by (2.3). A similar treatment yields an analogous result for the Stokes line arg $z = -\pi(1-\kappa)$.

It is worth pointing out that other exponentially small series are present in the expansion of ${}_1\Psi_q(-z)$ resulting from the terms $R_m(Z_n)$, $n \ge 2$. These switch on in a similar manner to (4.15) across the rays where $\arg(Z_n e^{-\pi i/\mu}) = \omega(\theta) - \pi\delta_n/\kappa = 0$; that is, on the rays $\theta = \pi(1-\kappa) + \pi\delta_n$ for n values corresponding to $\delta_n \le \kappa$. Although each of these additional series is of magnitude e^{-X} on these latter rays, they are subdominant with respect to $R_m(Z_1)$. This situation may be contrasted with that occurring in the gamma function $\Gamma(z)$, where for large $|z|$ an *infinite number of exponentially small terms of decreasing magnitude* switch on across the Stokes lines arg $z = \pm\frac{1}{2}\pi$; see [17, §6.4] for details.

4.2 Expansion of ${}_1\Psi_q(-z)$ on the Stokes line arg $z = \pi(1-\kappa)$, $0 < \kappa < 1$

We now obtain the central result of the paper, namely the expansion of ${}_1\Psi_q(-z)$ on the Stokes line $\arg z = \pi(1-\kappa)$, $0<\kappa<1$. On this ray we have $\omega(\theta)=0$ and $Z = Xe^{\pi i(1-\kappa)/\kappa}$. From (4.8), (4.13) and (4.14) with $n=1$ we obtain, upon neglecting exponentially small terms,

$$P_m(|z|e^{\pi i(1-\kappa)}) = (Xe^{-\pi i})^{\vartheta} e^{-X} \left\{ \sum_{j=0}^{M-1} (-)^j A_j X^{-j}\, T_{\nu-j}(\mu; Xe^{\pi i/\mu}) + O(X^{-M}) \right\}. \tag{4.17}$$

From (A.12) and (A.4), the expansion of the terminant function $T_{\nu-j}(\mu; Xe^{\pi i/\mu})$ on its Stokes line for $\nu \sim X \to +\infty$ is given by

$$T_{\nu-j}(\mu; Xe^{\pi i/\mu}) = \frac{1}{2} - \frac{i}{\sqrt{2\pi X}} \left\{ \sum_{k=0}^{N-1} g_{2k}(j)\, \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} (\tfrac{1}{2}X)^{-k} + O(X^{-N}) \right\} \tag{4.18}$$

for $j = 0, 1, 2, \ldots$, where N is a positive integer,

$$\begin{aligned}
g_0(j) &= \tfrac{1}{6} - \gamma_j + \tfrac{1}{2}\mu, \qquad \gamma_j := \nu - j - X,\\
g_2(j) &= \tfrac{1}{12}(\tfrac{1}{45} - \gamma_j + 3\gamma_j^2 - 2\gamma_j^3) + \tfrac{1}{4}\mu(\gamma_j^2 - \gamma_j + \tfrac{1}{6}) + \tfrac{1}{12}\mu^2(\tfrac{1}{2} - \gamma_j),\\
g_4(j) &= \tfrac{1}{120}(-\tfrac{65}{1512} - \tfrac{5}{36}\gamma_j + \tfrac{5}{2}\gamma_j^2 - \tfrac{50}{9}\gamma_j^3 + \tfrac{25}{6}\gamma_j^4 - \gamma_j^5) + \tfrac{1}{48}\mu(\tfrac{1}{36} - \gamma_j + \tfrac{10}{3}\gamma_j^2 - \tfrac{10}{3}\gamma_j^3 + \gamma_j^4)\\
&\quad + \tfrac{1}{72}\mu^2(\tfrac{1}{4} - \tfrac{5}{3}\gamma_j + \tfrac{5}{2}\gamma_j^2 - \gamma_j^3) + \tfrac{1}{720}\mu^4(-\tfrac{5}{6} + \gamma_j), \ldots ,
\end{aligned}$$

and we have introduced an argument in the coefficients g_{2k} to denote their dependence on the index j. Then, substitution of this expansion into (4.17), where we set $M = N$ for convenience, yields

$$P_m(|z|e^{\pi i(1-\kappa)}) = (Xe^{-\pi i})^{\vartheta} e^{-X} \left\{ \sum_{j=0}^{M-1} (-)^j \left\{ \frac{1}{2}A_j - \frac{iB_j}{\sqrt{2\pi X}} \right\} X^{-j} + O(X^{-M}) \right\} \tag{4.19}$$

with the coefficients B_j specified by

$$B_j = \sum_{k=0}^{j} (-2)^k \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\, A_{j-k}\, g_{2k}(j-k). \tag{4.20}$$

With the first expansion in (4.19) involving the coefficients A_j identified as $\frac{1}{2}E_{1,q}(ze^{-\pi i})$ by (2.3), we can then define the formal exponential expansion

$$\mathcal{E}_{1,q}(ze^{-\pi i}) := E_{1,q}(ze^{-\pi i}) - \frac{2i}{\sqrt{2\pi X}} (Xe^{-\pi i})^{\vartheta} e^{-X} \sum_{j=0}^{\infty} (-)^j B_j X^{-j} \tag{4.21}$$

on $\arg z = \pi(1-\kappa)$. Then, with the truncation index m in the algebraic expansion chosen according to (4.6), we finally obtain from (4.4) the result

$${}_1\Psi_q(-z) \sim \tfrac{1}{2}\mathcal{E}_{1,q}(ze^{-\pi i}) + H^o_{1,q}(z) \qquad (\arg z = \pi(1-\kappa);\ 0<\kappa<1) \tag{4.22}$$

as $|z| \to \infty$, where the superscript o signifies that the algebraic expansion in (4.4) is optimally truncated. This completes the proof of (2.10) in the case $p = 1$, $q \geq 0$ with $0 < \kappa < 1$. A similar treatment shows that (4.22) holds on the Stokes line $\arg z = -\pi(1-\kappa)$ when $e^{-\pi i}$ is replaced by $e^{\pi i}$ and $-i$ by i in (4.21).

For the function ${}_1\Psi_q(z)$, we can replace θ ($=\arg z$) by $\pi - \theta$ to find from (4.22) the expansion on the Stokes line $\arg z = \pi\kappa$

$${}_1\Psi_q(|z|e^{\pi i\kappa}) \sim \tfrac{1}{2}\mathcal{E}_{1,q}(z) + H^o_{1,q}(ze^{-\pi i}) \qquad (0<\kappa<1) \tag{4.23}$$

with

$$\mathcal{E}_{1,q}(z)=E_{1,q}(z)-\frac{2i}{\sqrt{2\pi X}}(Xe^{-\pi i})^{\vartheta}e^{-X}\sum_{j=0}^{\infty}(-)^jB_jX^{-j},\quad X=\kappa(h|z|)^{1/\kappa},\tag{4.24}$$

where $E_{1,q}(z)$ is defined by (2.3).

4.3 Expansion of ${}_1\Psi_q(-z)$ on the Stokes line $\arg z=0$ when $\kappa=1$

When $\alpha=\Sigma\beta_r$, the parameter $\kappa=1$ and the Stokes line for ${}_1\Psi_q(-z)$ is the positive real axis. Following the discussion surrounding (4.14), the expansion of $P_m(z)$ in this case is controlled by the terms in (4.8) corresponding to $n=1$ and $n=2q$. Then

$$P_m(z)\sim\Omega_1R_m(Z_1)+\Omega_{2q}R_m(Z_{2q})\qquad(z\to+\infty)$$

$$\sim(Xe^{-\pi i})^{\vartheta}e^{-X}\sum_{j=0}^{\infty}(-)^jA_jX^{-j}T_{\nu-j}(\mu;Xe^{\pi i/\mu})+(-)^qe^{-2\pi i\Sigma b_r}(Xe^{-\pi i(1+2\Sigma\beta_r)})^{\vartheta}e^{-Xe^{-2\pi i\Sigma\beta_r}}$$

$$\times\sum_{j=0}^{\infty}(-)^jA_j(Xe^{-2\pi i\Sigma\beta_r})^{-j}T_{\upsilon-j}(\mu;Xe^{-\pi i/\mu})$$

upon inserting the values of Ω_1 and Ω_{2q} from (4.7) and using (4.13). Employing the connection formula given in (A.13)

$$T_{\nu}(\mu;Xe^{-\pi i/\mu})=e^{2\pi i\nu/\mu}\exp\left\{-X+Xe^{-2\pi i/\mu}\right\}\left\{T_{\nu}(\mu;Xe^{\pi i/\mu})-1\right\}\qquad(X>0)$$

and recalling that $\mu=1/\Sigma\beta_r$ when $\kappa=1$, we find after some straightforward algebra, combined with use of the expansion in (4.18), that

$$P_m(z)\sim X^{\vartheta}e^{-X}\sum_{j=0}^{\infty}(-)^jA_jX^{-j}\left\{e^{-\pi i\vartheta}T_{\nu-j}(\mu;Xe^{\pi i/\mu})-e^{\pi i\vartheta}\{T_{\nu-j}(\mu;Xe^{\pi i/\mu})-1\}\right\}$$

$$\sim X^{\vartheta}e^{-X}\left\{\cos\pi\vartheta\sum_{j=0}^{\infty}(-)^jA_jX^{-j}-\frac{2\sin\pi\vartheta}{\sqrt{2\pi X}}\sum_{j=0}^{\infty}(-)^jB_jX^{-j}\right\}\tag{4.25}$$

as $X=hz\to+\infty$, where the coefficients B_j are given in (4.20). The expansion of ${}_1\Psi_q(-z)$ as $z\to+\infty$ when $\kappa=1$ is then given by (4.4) and (4.25).

5. Numerical examples

In this section we present some numerical examples to demonstrate the accuracy of the results obtained in Section 4. As a first example, let us consider the function with $p=1$, $q=0$ discussed in Section 3 by the saddle-point method, namely

$${}_1\Psi_0(z)=\sum_{n=0}^{\infty}\Gamma(\alpha n+1)\frac{z^n}{n!}\qquad(0<\alpha<1)\tag{5.1}$$

with $a=1$. From (1.3) we have the parameters $\kappa=1-\alpha$, $h=\alpha^{\alpha}$, $\vartheta=\frac{1}{2}$ and $A_0=(2\pi\alpha)^{1/2}/\kappa$. With $z=xe^{\pi i\kappa}$, $X=\kappa(hx)^{1/\kappa}$, $x>0$, the algebraic expansion is given in (3.4) and, from (4.24), the exponential expansion is

$$\mathcal{E}_{1,0}(z)=iX^{1/2}e^{-X}\sum_{j=0}^{\infty}(-)^jA_jX^{-j}-\frac{2e^{-X}}{\sqrt{2\pi}}\sum_{j=0}^{\infty}(-)^jB_jX^{-j},$$

where the coefficients A_j and B_j are given in (3.8) and (4.20) respectively.

If we now define

$$F(x) := {}_1\Psi_0(xe^{\pi i\kappa}) + \frac{1}{\alpha}\sum_{k=0}^{m-1}\frac{\Gamma((k+1)/a)}{k!}x^{-(k+1)/\alpha}, \tag{5.2}$$

where m is chosen to be the optimal truncation index $\mu m \sim X$, with $\mu = (1-\alpha)/\alpha$, then it follows from (4.23), (4.24) and (2.4) that

$$F(x) \sim \frac{1}{2}iX^{1/2}e^{-X}\sum_{j=0}^{\infty}(-)^j A_j X^{-j} - \frac{e^{-X}}{\sqrt{2\pi}}\sum_{j=0}^{\infty}(-)^j B_j X^{-j} \qquad (x \to +\infty). \tag{5.3}$$

In the calculation of the coefficients B_j we use the parameters $\nu = \frac{1}{2} + (m+1)\mu$ and $\gamma_j = \nu - j - X$. The expansion obtained from the saddle-point analysis in (3.6) yields

$$F(x) \sim \frac{1}{2}iX^{1/2}e^{-X}\sum_{j=0}^{\infty}(-)^j A_j X^{-j} + e^{-X}\sum_{j=0}^{\infty}(-)^j C_j X^{-j}, \tag{5.4}$$

where

$$C_0 = \frac{2-\alpha}{3\kappa}, \qquad C_1 = \frac{4(2-\alpha)}{135\alpha\kappa}(1-\alpha-2\alpha^2), \ldots\ .$$

It is seen that the imaginary parts of $F(x)$ in the expansions (5.3) and (5.4) agree, but that the real parts differ.

Table 1: Values of the absolute relative error in the computation of $F(x)$ in (5.2) using an optimal truncation $m \sim X$ of the algebraic expansion for different summation index j.

	$\alpha = 0.45,\ x = 10$ $m = 14$ $\mathrm{Re}\,F = -6.307298(-9)$ $\mathrm{Im}\,F = +4.409535(-9)$		$\alpha = 0.55,\ x = 7.50$ $m = 22$ $\mathrm{Re}\,F = -2.855966(-9)$ $\mathrm{Im}\,F = +4.692973(-9)$		$\alpha = 0.50,\ x = 8$ $m = 15$ $\mathrm{Re}\,F = -2.633405(-8)$ $\mathrm{Im}\,F = +7.978536(-7)$	
j	Error Re F	Error Im F	Error Re F	Error Im F	Error Re F	Error Im F
0	5.971(−3)	7.435(−4)	2.962(−3)	5.615(−4)	7.243(−3)	0
1	2.816(−5)	1.935(−5)	8.035(−5)	1.402(−5)	7.379(−5)	0
2	4.266(−6)	1.195(−6)	4.240(−6)	9.035(−7)	2.815(−6)	0

In Table 1 we show the absolute relative errors[5] in the computation of $\mathrm{Re}\,F(x)$ and $\mathrm{Im}\,F(x)$ from the expansion (5.3) for different values of α, x and truncation index $j \leq 2$. The evaluation of $F(x)$ was obtained by high-precision computation of the series in (5.1) and the optimal truncation index m of the algebraic expansion was determined by inspection. The expansion

$$\mathrm{Re}\,F(x) \sim -\frac{e^{-X}}{\sqrt{2\pi}}\sum_{j=0}^{\infty}(-)^j B_j X^{-j}$$

obtained from (5.3) and that for $\mathrm{Im}\,F(x)$ are found to agree well with the exact values. However, the value of $\mathrm{Re}\,F(x)$ obtained from (5.4) does not yield agreement; for example, when $\alpha = 0.50$, $x = 8$, the real part of (5.4) yields the value $+1.12535\times 10^{-7}$, which is not

[5]In Tables 1 and 2 we write the values as $x(y)$ instead of $x \times 10^y$.

only of incorrect magnitude but also of the wrong sign. This indicates that the saddle-point calculation in Section 3 has not captured all of the exponentially small contribution. We remark that the error in $\operatorname{Im} F(x)$ when $\alpha = \frac{1}{2}$ $(\kappa = \frac{1}{2})$ is zero since, from (5.3) and the fact that the coefficients $A_j = 0$ $(j \geq 1)$, the expression $\operatorname{Im} F(x) = \frac{1}{2} A_0 X^{1/2} e^{-X}$, $X = x^2/4$ is *exact*. This follows from a simple calculation using the duplication formula for the gamma function to show that

$$\sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}n+1)}{n!}(ix)^n = {}_1F_1(1; \tfrac{1}{2}; -X^2) + \tfrac{1}{2} i A_0 X^{1/2} e^{-X},$$

where ${}_1F_1$ denotes the confluent hypergeometric function.

Our second example is the function with $p = q = 1$, $\alpha = \beta$

$${}_1\Psi_1(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha n + a)}{\Gamma(\alpha n + b)} \frac{z^n}{n!} \qquad (a \neq b),$$

corresponding to the parameters $\kappa = 1$, $h = 1$ and $\vartheta = a - b$, for which the Stokes line is the negative real z-axis. This subclass of Wright functions has recently emerged in probability theory in [15]. With $z = xe^{\pi i}$, $X = x$, we obtain from (4.4) and (4.25) the expansion

$$F(x) := {}_1\Psi_1(-x) - H_{1,1}(x) \sim X^{\vartheta} e^{-X} \left\{ \cos \pi\vartheta \sum_{j=0}^{\infty} (-)^j A_j X^{-j} - \frac{2 \sin \pi\vartheta}{\sqrt{2\pi X}} \sum_{j=0}^{\infty} (-)^j B_j X^{-j} \right\} \tag{5.5}$$

as $x \to +\infty$, where the algebraic expansion $H_{1,1}(x)$ is

$$H_{1,1}(x) = \frac{1}{\alpha} \sum_{k=0}^{m-1} \frac{(-)^k}{k!} \frac{\Gamma((k+a)/\alpha)}{\Gamma(b-a-k)} x^{-(k+a)/\alpha}$$

and m is chosen to be the optimal truncation index.

When $\vartheta = N$ an integer, the algebraic expansion $H_{1,1}(x)$ consists of a finite number of terms: if $N \geq 1$ we have $H_{1,1}(x) \equiv 0$ and if $N \leq -1$ then $H_{1,1}(x)$ consists of $m = |N|$ terms. In this case the truncation index m is no longer optimal and the parameter ν in (4.2) is then finite as $x \to +\infty$. As a consequence, the analysis presented in Section 4 based on the terminant function $T_\nu(\mu; z)$ of large order ν and variable z is no longer valid. It can be shown by means of the approach employed in [12, §5] that

$$F(x) \sim X^{\vartheta} e^{-X} \cos \pi\vartheta \sum_{j=0}^{\infty} (-)^j A_j X^{-j} \qquad (\vartheta = N,\ x \to +\infty). \tag{5.6}$$

These details are omitted here but will be presented elsewhere in relation to the confluent hypergeometric functions.[6]

In Table 2 we show values of the absolute relative error in the computation of $F(x)$ from (5.5) for different x using the terms corresponding to $j \leq 1$. The coefficients A_0 and A_1 are obtained from (2.2); higher-order coefficients can be obtained using the algorithm described in [12, Appendix]. The last entry has $\vartheta = -1$ so that the algebraic expansion consists of a single term $(m = 1)$ and the expansion of $F(x)$ is given by (5.6).

For our final example, we take $p = 1$, $q = 2$ and consider the function

$${}_1\Psi_2(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{4}n + a)}{\Gamma(\frac{3}{8}n + b_1)\Gamma(\frac{1}{8}n + b_2)} \frac{z^n}{n!} \qquad (\kappa = \tfrac{3}{4}),$$

[6] When $\vartheta = N \geq 1$, it can be shown that ${}_1\Psi_1(z)$ reduces to a polynomial in z of degree N multiplied by e^z.

Table 2: Values of the absolute relative error and the truncation index m in the computation of $F(x)$ in (5.5) for different x. The exponential expansions are evaluated with index $j \leq 1$.

	$\alpha = 0.50$ $a = 0.75,\ b = 0.50$		$\alpha = 0.75,\ a = 0.50,\ b = 0.75$	
x	m	Error	m	Error
10	4	9.920(−4)	8	1.290(−3)
20	9	1.272(−4)	15	2.633(−4)
25	12	1.059(−5)	19	1.040(−4)
50	24	8.228(−6)	38	4.619(−5)
	$\alpha = 1.50,\ a = 1,\ b = 0.75$		$\alpha = 0.75,\ a = 0.50,\ b = 1.50$	
x	m	Error	m	Error
10	14	1.110(−4)	1	3.765(−3)
20	29	3.364(−5)	1	1.015(−3)
25	37	2.146(−5)	1	6.607(−4)
50	74	6.210(−6)	1	1.171(−4)

which is associated with the parameters $\vartheta = a - b_1 - b_2 + \frac{1}{2}$, $h = (\frac{3}{4})^{\frac{3}{4}}(\frac{3}{8})^{-\frac{3}{8}}(\frac{1}{8})^{-\frac{1}{8}}$ and

$$A_0 = (2\pi)^{-\frac{1}{2}}\kappa^{-\frac{1}{2}-\vartheta}(\tfrac{3}{4})^{a-\frac{1}{2}}(\tfrac{3}{8})^{\frac{1}{2}-b_1}(\tfrac{1}{8})^{\frac{1}{2}-b_2}.$$

The Stokes line in the upper half-plane is $\arg z = \frac{3}{4}\pi$. The algebraic expansion is, from (2.6),

$$H_{1,2}(ze^{-\pi i}) = \frac{4}{3}\sum_{k=0}^{\infty}\frac{(-)^k}{k!}\,\frac{\Gamma(\frac{4}{3}(k+a))(ze^{-\pi i})^{-4(k+a)/3}}{\Gamma(b_1 - \frac{1}{2}k - \frac{1}{2}a)\Gamma(b_2 - \frac{1}{6}k - \frac{1}{6}a)}.$$

From (4.16) (with θ replaced by $\pi - \theta$), the error-function smoothing factor, which multiplies the exponential expansion $E_{1,2}(z)$ in the neighbourhood of the Stokes line $\arg z = \pi\kappa = \frac{3}{4}\pi$, is given by

$$\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left\{(\theta - \pi\kappa)\left(\frac{(h|z|)^{1/\kappa}}{2\kappa}\right)^{1/2}\right\}. \tag{5.7}$$

In the sense of increasing $\arg z$, the exponential expansion $E_{1,2}(z)$, which is dominant in the sector $|\arg z| < \frac{3}{8}\pi$, switches off smoothly across the Stokes line $\arg z = \frac{3}{4}\pi$ to leave the algebraic expansion in the sector $\frac{3}{4}\pi < \arg z \leq \pi$. A similar transition occurs in the lower half-plane on the Stokes line $\arg z = -\frac{3}{4}\pi$.

To demonstrate this assertion, we calculate the so-called Stokes multiplier $S(\theta)$, which for the transition across $\arg z = \frac{3}{4}\pi$ at fixed $|z|$, is *defined* by

$${}_1\Psi_2(z) = H^o_{1,2}(ze^{-\pi i}) + A_0 Z^\vartheta e^Z\, S(\theta), \qquad Z = \kappa(hz)^{1/\kappa},$$

where the superscript o denotes that the algebraic expansion is *optimally truncated*. Then we have

$$S(\theta) := \frac{{}_1\Psi_2(z) - H^o_{1,2}(ze^{-\pi i})}{A_0 Z^\vartheta e^Z}. \tag{5.8}$$

According to Section 4, $S(\theta)$ should have the approximate functional form given by (5.7) as $|z| \to \infty$ in the neighbourhood of the Stokes line $\arg z = \frac{3}{4}\pi$. In Table 3, we show the values of the real part[7] of the Stokes multiplier $S(\theta)$ in the neighbourhood of $\arg z = \frac{3}{4}\pi$

[7]The Stokes multiplier $S(\theta)$ has a small imaginary part that we do not show.

when $z = 10e^{i\theta}$ and $a = \frac{1}{3}$, $b_1 = \frac{1}{4}$, $b_2 = \frac{3}{4}$. It is seen that the computed values of the real part of $S(\theta)$ follow closely the values predicted by (5.7).

Table 3: Values of the real part of the Stokes multiplier $S(\theta)$ compared with the approximate value obtained from (5.7) for ${}_1\Psi_2(z)$ across the Stokes line $\arg z = \frac{3}{4}\pi$ when $|z| = 10$.

θ/π	$\mathrm{Re}\, S(\theta)$	Approx $S(\theta)$	θ/π	$\mathrm{Re}\, S(\theta)$	Approx $S(\theta)$
0.60	0.9949	0.9996	0.75	0.4960	0.5000
0.65	0.9847	0.9867	0.76	0.4054	0.4123
0.70	0.8699	0.8661	0.78	0.2415	0.2531
0.72	0.7499	0.7469	0.80	0.1207	0.1339
0.74	0.5865	0.5877	0.85	0.0126	0.0133

7. Concluding remarks

We have established the form of the exponentially small expansion of the Wright function ${}_1\Psi_q(z)$, with $q \geq 0$, as $|z| \to \infty$ on the Stokes lines $\arg z = \pm\pi\kappa$, $0 < \kappa \leq 1$. In addition, the smooth transition of the Stokes multiplier associated with the exponentially expansion when $\kappa < 1$ has been shown to follow the familiar error-function smoothing law first obtained in [1]. The case of ${}_p\Psi_q(z)$ with $p \geq 1$, $q = 0$ has been considered in [14] but was subject to the condition $\alpha_r = \alpha$ $(1 \leq r \leq p)$, which represents an awkward restriction. A special case of this last function was established earlier in [10] in connection with the solution of a certain ordinary differential equation of order $n \geq 2$.

There remains the general case of ${}_p\Psi_q(z)$ to consider. It is hoped that the developments presented in this work will stimulate further interest and research in this area of hypergeometric-like functions.

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Appendix A: The terminant function $T_\nu(\mu; z)$

Let the parameters μ and ν satisfy $\mu > 0$, $\mathrm{Re}\,\nu > 0$ and put $z = xe^{i\phi}$, $x > 0$. The *terminant function* $T_\nu(\mu; z)$ employed in Section 4 is defined by the Mellin-Barnes integral

$$T_\nu(\mu; z) := \frac{\mu}{2i}\,(ze^{-\pi i/\mu})^{-\nu} e^{ze^{-\pi i/\mu}}\, J_\nu(\mu; z), \tag{A.1}$$

where

$$J_\nu(\mu; z) := \frac{i}{2\pi}\int_{-c-\infty i}^{-c+\infty i} \Gamma(\mu s + \nu)\,\frac{z^{-\mu s}}{\sin \pi s}\,ds \qquad (0 < c < 1) \tag{A.2}$$

for $|\arg z| < \frac{1}{2}\pi + \pi/\mu$. The integration path in (A.2) lies to the right of the poles of the gamma function; that is, $-\mu c + \mathrm{Re}\,\nu > 0$. In order to examine the behaviour of this function for large x and ν, we convert the above integral into a Laplace integral. To achieve this we replace the gamma function by its Euler integral representation (provided $-\mu c + \mathrm{Re}\,\nu > 0$) combined with use of the result [17, p. 91]

$$\frac{1}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \frac{\pi\chi^{-s}}{\sin \pi s}\,ds = -\frac{\chi}{1+\chi} \qquad (0 < c < 1;\ |\arg \chi| < \pi).$$

Upon reversal of the order of integration, we obtain

$$J_\nu(\mu;z)=\frac{1}{\pi}\int_0^\infty \frac{t^{\nu-1}e^{-t}}{1+(t/z)^\mu}dt=\frac{x^\nu}{\pi}\int_0^\infty \frac{\tau^{\nu-1}\exp(-x\tau)}{1+\tau^\mu e^{-i\mu\phi}}\,d\tau \qquad (|\phi|<\pi/\mu), \tag{A.3}$$

where we have introduced the change of variable $\tau=t/x$.

A.1 The behaviour of $T_\nu(\mu;z)$ for large $\nu\sim x$

We now consider the leading behaviour of $J_\nu(\mu;z)$ as $x\to+\infty$ when it is supposed that

$$\nu=x+\gamma, \tag{A.4}$$

where $|\gamma|$ is bounded. Then we have

$$J_\nu(\mu;z)=\frac{x^\nu}{\pi}\int_0^\infty \exp\left[-x(\tau-\log\tau)\right]\frac{\tau^{\gamma-1}}{1+\tau^\mu e^{-i\mu\phi}}\,d\tau,$$

which is associated with a saddle point at $\tau=1$ and a pole at $\tau_p=e^{i(\phi\mp\pi/\mu)}$. As $\phi\to\pm\pi/\mu$, the pole and saddle point become coincident. A straightforward application of Laplace's method to the above integral shows that

$$J_\nu(\mu;z)=\sqrt{\frac{2}{\pi x}}\,\frac{x^\nu e^{-x}}{1+e^{-i\mu\phi}}\,\{1+O(x^{-1})\} \tag{A.5}$$

uniformly as $x\to+\infty$ in the sector $|\phi|\le\pi/\mu-\epsilon$, $\epsilon>0$. The approximation in (A.5) ceases to be valid in the neighbourhood of $\phi=\pm\pi/\mu$. A uniform approximation for $J_\nu(\mu;z)$, which is valid in a sector enclosing the rays $\phi=\pm\pi/\mu$, can be obtained in a manner similar to that given by Olver [7] in the case $\mu=1$; see also [17, pp. 262–265] for a summary.

We make the standard quadratic transformation

$$\tfrac{1}{2}w^2=\tau-\log\tau-1, \qquad \frac{d\tau}{dw}=\frac{w\tau}{\tau-1}, \tag{A.6}$$

so that the saddle point $\tau=1$ corresponds to $w=0$. The branch of w is chosen such that $w\sim\tau-1$ as $\tau\to1$. The pole at $\tau_p=e^{i(\phi-\pi/\mu)}$ then corresponds to the point $w=w_0$, where $w_0=ic(\phi)$ and $c(\phi)$ is defined by

$$\tfrac{1}{2}c^2(\phi)=1+i(\phi-\pi/\mu)-e^{i(\phi-\pi/\mu)}.$$

With the branch for w chosen as above, this yields

$$c(\phi)\simeq\phi-\pi/\mu \qquad (\phi\simeq\pi/\mu). \tag{A.7}$$

The integral in (A.3) is valid in $|\phi|<\pi/\mu$. However, it can be analytically continued to the sector $\pi/\mu\le\phi<2\pi/\mu$ provided the integration path is deformed to pass above the pole τ_p. In terms of the new variable w, (A.3) then becomes

$$J_\nu(\mu;z)=\frac{x^\nu e^{-x}}{\pi\mu}\int_{-\infty}^\infty e^{-\frac{1}{2}xw^2}f(w)\,dw, \qquad f(w)=\frac{\mu\tau^{\gamma-1}}{(1+\tau^\mu e^{-i\mu\phi})}\,\frac{d\tau}{dw} \tag{A.8}$$

and the integration path is indented to pass *above* the point $w=w_0$ when $\phi\ge\pi/\mu$. The function $f(w)$ has a simple pole at $w=w_0$ and so may be written in the form

$$f(w)=\frac{C}{w-w_0}+g(w), \tag{A.9}$$

where $g(w)$ is analytic at $w = w_0$ and $C = \lim_{w\to w_0}(w - w_0)f(w)$. A straightforward application of l'Hospital's rule as $w \to w_0$, $\tau \to \tau_p$ shows that $C = -\exp[i\gamma(\phi - \pi/\mu)]$.

Substitution of the above form for $f(w)$ into (A.8) then yields

$$x^{-\nu}e^x J_\nu(\mu; z) = \frac{1}{\pi\mu}\int_{-\infty}^{\infty} e^{-\frac{1}{2}xw^2}\left\{\frac{C}{w - w_0} + g(w)\right\}dw$$
$$= -\frac{iC}{\mu}e^{\frac{1}{2}xc^2(\phi)}\,\mathrm{erfc}\,[-c(\phi)(\tfrac{1}{2}x)^{\frac{1}{2}}] + O(x^{-\frac{1}{2}}),$$

where the first part of the integral has been evaluated in terms of the complementary error function erfc. Since $\exp[\frac{1}{2}xc^2(\phi)] = \exp[x + i(\phi - \pi/\mu)\,x - ze^{-\pi i/\mu}]$, we finally obtain using (A.4) that

$$J_\nu(\mu; z) = \frac{i}{\mu}(ze^{-\pi i/\mu})^\nu e^{-ze^{-\pi i/\mu}}\,\{1 + \mathrm{erf}\,[c(\phi)(\tfrac{1}{2}x)^{\frac{1}{2}}]\} + O(x^{\nu-\frac{1}{2}}e^{-x}) \tag{A.10}$$

as $|z| \to \infty$ in the sector $\epsilon \le \phi \le 2\pi/\mu - \epsilon$ when $\nu \simeq x$. A conjugate behaviour holds in the sector $-2\pi/\mu + \epsilon \le \phi \le -\epsilon$.

From (A.1), (A.5) and (A.10) we then find the behaviour

$$T_\nu(\mu; z) = \begin{cases} -\dfrac{i\mu}{\sqrt{2\pi x}}\,\dfrac{e^{(\pi/\mu-\phi)i\nu}}{1 + e^{-i\mu\phi}}\,e^{-x+ze^{-\pi i/\mu}}\{1 + O(z^{-1})\} & |\phi| \le \pi/\mu - \epsilon \\ \\ \frac{1}{2} + \frac{1}{2}\mathrm{erf}\,[c(\phi)(\frac{1}{2}x)^{\frac{1}{2}}] + O(x^{-\frac{1}{2}}e^{-\frac{1}{2}xc^2(\phi)}) & \epsilon \le \phi \le 2\pi/\mu - \epsilon \end{cases} \tag{A.11}$$

as $x \to +\infty$ when $\nu \simeq x$. In the case $\mu = 1$, this agrees with that found in [7], where it was established that the behaviour in the second part of (A.11) holds in the wider sector $-\pi + \epsilon \le \phi \le 3\pi - \epsilon$.

A.2 The expansion of $T_\nu(\mu; z)$ on the Stokes line $\arg z = \pi/\mu$

From (A.7), we have on the Stokes line $\phi = \pi/\mu$ that $c(\phi) = 0$ and $w_0 = 0$. Accordingly the second part of (A.11) reduces to

$$T_\nu(\mu; xe^{\pi i/\mu}) = \tfrac{1}{2} + O(x^{-\frac{1}{2}})$$

as $x \to +\infty$. To derive the expansion of $T_\nu(\mu; xe^{\pi i/\mu})$ when $\nu \sim x$, we require the expansion of $g(w)$ in (A.9) in ascending powers of w. Reversion of (A.6) yields

$$\tau = 1 + w + \tfrac{1}{3}w^2 + \tfrac{1}{36}w^3 - \tfrac{1}{270}w^4 + \tfrac{1}{4320}w^5 + \cdots\ .$$

Then, when $\phi = \pi/\mu$, we have in the neighbourhood of $w = 0$

$$g(w) = \frac{\mu\tau^{\gamma-1}}{(1 - \tau^\mu)}\,\frac{d\tau}{dw} + \frac{1}{w} = \sum_{k=0}^{2N-1} g_k w^k + w^{2N}G_{2N}(w)$$

for positive integer N, where[8]

$$g_0 = \tfrac{1}{6} - \gamma + \tfrac{1}{2}\mu, \quad g_2 = \tfrac{1}{12}(\tfrac{1}{45} - \gamma + 3\gamma^2 - 2\gamma^3) + \tfrac{1}{4}\mu(\gamma^2 - \gamma + \tfrac{1}{6}) + \tfrac{1}{12}\mu^2(\tfrac{1}{2} - \gamma),$$
$$g_4 = \tfrac{1}{120}(-\tfrac{65}{1512} - \tfrac{5}{36}\gamma + \tfrac{5}{2}\gamma^2 - \tfrac{50}{9}\gamma^3 + \tfrac{25}{6}\gamma^4 - \gamma^5) + \tfrac{1}{48}\mu(\tfrac{1}{36} - \gamma + \tfrac{10}{3}\gamma^2 - \tfrac{10}{3}\gamma^3 + \gamma^4)$$
$$+ \tfrac{1}{72}\mu^2(\tfrac{1}{4} - \tfrac{5}{3}\gamma + \tfrac{5}{2}\gamma^2 - \gamma^3) + \tfrac{1}{720}\mu^4(-\tfrac{5}{6} + \gamma), \ldots\ .$$

[8] We present only the even-order coefficients since those of odd order do not contribute to the expansion of $T_\nu(\mu; z)$. In addition, it may be noted that $\mu w/(\tau^\mu - 1) = 1 + \sum_{k=0}^\infty D_k(\mu)w^k$ near $w = 0$, where $D_1(\mu) = \frac{1}{6} - \frac{1}{2}\mu$ and $D_k(\mu)$ $(k \ge 2)$ involve polynomials in μ^2 of degree $\lfloor k/2 \rfloor$.

From (A.1), (A.2) and (A.8), we then obtain the expansion on the Stokes line arg $z = \pi/\mu$

$$\begin{aligned} T_\nu(\mu; xe^{\pi i/\mu}) &= \frac{1}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}xw^2}\biggl\{\sum_{k=0}^{2N-1} g_k w^k + w^{2N} G_{2N}(w)\biggr\} dw \\ &= \frac{1}{2} - \frac{i}{\sqrt{2\pi x}}\biggl\{\sum_{k=0}^{N-1} g_{2k}(\tfrac{1}{2})_k(\tfrac{1}{2}x)^{-k} + O(x^{-N})\biggr\} \end{aligned} \qquad \text{(A.12)}$$

as $x \to +\infty$ when $\nu \sim x$. The analysis of the remainder term in (A.12) resulting from the integral of $G_{2N}(w)$ follows closely that given in [8, Section 4] in the case $\mu = 1$ and is not repeated here.

A.4 The connection formula for $T_\nu(\mu; xe^{\pm\pi i/\mu})$

The connection formula between $T_\nu(\mu; xe^{\pm\pi i/\mu})$ can be determined by use of (A.1) to yield

$$T_\nu(\mu; xe^{\pi i/\mu}) = x^{-\nu}e^x\,\frac{\mu}{4\pi}\int_{-c-\infty i}^{-c+\infty i} \Gamma(\mu s + \nu)\frac{x^{-\mu s}e^{-\pi is}}{\sin \pi s} ds$$

and

$$e^{-2\pi i\nu/\mu} T_\nu(\mu; xe^{-\pi i/\mu}) = x^{-\nu}e^{xe^{-2\pi i/\mu}}\,\frac{\mu}{4\pi}\int_{-c-\infty i}^{-c+\infty i} \Gamma(\mu s + \nu)\frac{x^{-\mu s}e^{\pi is}}{\sin \pi s} ds.$$

Thus

$$\begin{aligned} &e^{-2\pi i\nu/\mu}e^{x - xe^{-2\pi i/\mu}} T_\nu(\mu; xe^{-\pi i/\mu}) - T_\nu(\mu; xe^{\pi i/\mu}) \\ &\qquad = -\frac{\mu e^x}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \Gamma(\mu s + \nu) x^{-\mu s - \nu} ds = -1 \qquad (-\mu c + \mathrm{Re}\,(\nu) > 0) \end{aligned}$$

by means of the well-known Cahen-Mellin integral; see, for example, [17, p. 89]. Hence, we obtain the connection formula

$$T_\nu(\mu; xe^{-\pi i/\mu}) = e^{2\pi i\nu/\mu}e^{-x + xe^{-2\pi i/\mu}}\{T_\nu(\mu; xe^{\pi i/\mu}) - 1\}. \qquad \text{(A.13)}$$

When $\mu = 1$, (A.13) reduces to the connection formula given in [17, (6.2.45)].

Appendix B: Bounds for the remainder $R_{M,m}(Z_n)$

We consider bounds for the remainder function $R_{M,m}(Z_n)$ in (4.10) where, for simplicity in presentation, we remove inessential O(1) multiplicative factors and write

$$\mathcal{R}_{M,m}(Z_n) = \frac{Z_n^{-\mu a}}{2\pi i}\int_{-c+m-\infty i}^{-c+m+\infty i} \Gamma(\mu s + \xi)\,\hat{\sigma}_M(s)\,\frac{Z_n^{-\mu s}}{\sin \pi s} ds, \qquad \xi := \mu a + \vartheta - M, \quad \text{(B.1)}$$

where $0 < c < 1$ and $\hat{\sigma}_M(s) = O(1)$ as $|s| \to \infty$ in $|\arg s| < \pi$. A slight modification of Lemma 2.9 in [17, p. 75] using $\mu m \sim X$ as $X \to +\infty$ shows that

$$\mathcal{R}_{M,m}(Z_n) = O(X^{\vartheta - M}e^{-X}) \quad \text{in} \quad |\arg Z_n| \le \pi/\mu. \qquad \text{(B.2)}$$

In terms of $\omega(\theta)$ defined in (4.11), the sector $|\arg Z_n| \le \pi/\mu$ corresponds to the sector $-2\pi/\mu + \Delta_n \le \omega(\theta) \le \Delta_n$, where $\Delta_n = \pi\delta_n/\kappa$.

To extend the bound for $n = 1$ beyond $\omega(\theta) = 0$ into $\omega(\theta) > 0$, we follow the procedure described in [17, p. 244] and substitute the identity

$$\frac{1}{\sin \pi s} = \frac{e^{2\pi is}}{\sin \pi s} - 2ie^{\pi is}$$

into (B.1) to obtain

$$\mathcal{R}_{M,m}(Z_1) = \mathcal{R}_{M,m}(Z_1 e^{-2\pi i/\mu}) - \frac{1}{\pi} \int_{-c+m-\infty i}^{-c+m+\infty i} \Gamma(\mu s + \xi)\, \hat{\sigma}_M(s) (Z_1 e^{-\pi i/\mu})^{-\mu s} ds. \quad \text{(B.3)}$$

Application of Lemma 2.8 in [17, p. 72] (after bending back the integration path into a loop with endpoints passing to infinity in $\mathrm{Re}\, s < 0$) shows that the integral in (B.3) is $\mathrm{O}(X^{\vartheta - M} e^{-X})$ in $|\arg(Z_1 e^{-\pi i/\mu})| < \pi$; that is, from (4.12), in the sector given by $|\omega(\theta)| < \pi$. From (B.2), $\mathcal{R}_{M,m}(Z_1 e^{-2\pi i/\mu}) = O(X^{\vartheta - M} e^{-X})$ in $|\arg(Z_1 e^{-2\pi i/\mu})| < \pi/\mu$, which corresponds to the sector $0 < \omega(\theta) < 2\pi/\mu$.

Hence, we have

$$\mathcal{R}_{M,m}(Z_1) = O(X^{\vartheta - M} e^{-X}) + O(X^{\vartheta - M} e^{Z e^{-\pi i/\kappa}}) = O(X^{\vartheta - M} e^{Z e^{-\pi i/\kappa}}) \quad \text{(B.4)}$$

as $|z| \to \infty$ in the sector $0 < \omega(\theta) < \min\{\pi, 2\pi/\mu\}$.

References

[1] M. V. Berry, Uniform asymptotic smoothing of Stokes's discontinuities, Proc. Roy. Soc. London **A422** (1989) 7–21.

[2] B. L. J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, Compos. Math. **15** (1963) 239–341.

[3] R. B. Dingle, Asymptotic Expansions: Their Derivation and Interpretation, Academic Press, London, 1973.

[4] I. A. Ibragimov and Yu. V. Linnik, Independent and Stationary Sequences of Random Variables, Walters–Noordhoff, Groningen, 1971.

[5] Yu. V. Linnik, On stable probability laws with exponent less than one, Dokl. Akad. Nauk SSSR **94** (1954) 619–621 [in Russian].

[6] F. W. J. Olver, Asymptotics and Special functions, Academic Press, New York, 1974. Reprinted A. K. Peters, Massachusets, 1997.

[7] F. W. J. Olver, On Stokes' phenomenon and converging factors, In Proc. Int. Conf. on Asymptotic and Comp. Analysis (ed. R. Wong) (1990) 329–355, Marcel Dekker, New York.

[8] F. W. J. Olver, Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral, SIAM J. Math. Anal. **22** (1991) 1460–1474.

[9] R. B. Paris, Smoothing of the Stokes phenomenon using Mellin-Barnes integrals, J. Comput. Appl. Math. **41** (1992) 117–133.

[10] R. B. Paris, Smoothing of the Stokes phenomenon for high-order differential equations, Proc. Roy. Soc. London **A436** (1992) 165–186.

[11] R. B. Paris, Exponential asymptotics of the Mittag-Leffler function, Proc. Roy. Soc. London **A458** (2002) 3041–3052..

[12] R. B. Paris, Exponentially small expansions in the asymptotics of the Wright function, J. Comput. Appl. Math. **234** (2010) 488–504.

[13] R. B. Paris, Hadamard Expansions and Hyperasymptotic Evaluation, Cambridge University Press, Cambridge, 2011.

[14] R. B. Paris, Exponential smoothing of the Wright function, Technical Report MS 11:01, University of Abertay Dundee, 2011.

[15] R. B. Paris and V. Vinogradov, Refined local approximations for members of some Poisson-Tweedie EDMs 2013 [in preparation].

[16] R. B. Paris and A. D. Wood, Asymptotics of High Order Differential Equations, Pitman Research Notes in Mathematics, **129**, Longman Scientific and Technical, Harlow, 1986.

[17] R. B. Paris and D. Kaminski, Asymptotics and Mellin-Barnes Integrals, Cambridge University Press, Cambridge, 2001.

[18] A. V. Skorokhod, Asymptotic formulas for stable distribution laws, Dokl. Akad. Nauk SSSR **98** (1954) 731–734 [in Russian].

[19] L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.

[20] V. Vinogradov, R. B. Paris and O. Yanushkevichiene, New properties and representations for members of the power-variance family. I, Lith. Math. J. **52** (2012) 444–461.

[21] V. Vinogradov, R. B. Paris and O. Yanushkevichiene, New properties and representations for members of the power-variance family. II, Lith. Math. J. **53** (2013) 103–120.

[22] V. Vinogradov, R. B. Paris and O. Yanushkevichiene, The Zolotarev polynomials revisited, XXXI International Seminar on Stability Problems for Stochastic Models, pp. 68–70, Institute of Informatics Problems, Russian Academy of Sciences, Moscow, 2013.

[23] R. Wong and Y.-Q. Zhao, Smoothing of Stokes's discontinuity for the generalized Bessel function, Proc. Roy. Soc. London **A455** (1999) 1381–1400.

[24] R. Wong and Y.-Q. Zhao, Exponential asymptotics of the Mittag-Leffler function, Constr. Approx. **18** (2002) 355–385.

[25] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, J. Lond. Math. Soc. (Ser. 2) **10** (1935) 286–293.

[26] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, Proc. Lond. Math. Soc. (Ser. 2) **46** (1940) 389–408.

[27] V. M. Zolotarev, Expression of the density of a stable distribution with exponent α greater than one by means of a density with exponent $1/\alpha$, Dokl. Akad. Nauk SSSR **98** (1954) 735–738 [in Russian].

[28] V. M. Zolotarev, One-Dimensional Stable Distributions, Amer. Math. Soc., Providence, RI, 1986.